Technical Comments

Comments on "Theory of Ion Boundary Layer"

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THE continuum solutions presented in the paper, "Theory of Ion Boundary Layer,"[1] for flat plate and cylindrical probes contain basic inconsistencies. The difficulties are caused by the fact that the so-called equivalent one-dimensional situation is not an equivalent. A fundamental feature of the ion boundary layer problem is that the ion flux toward the wall varies from zero at the outer edge of the sheath to its value at the wall. In the equivalent one-dimensional solution this flux is required to be a constant across the sheath which leads to a violation of the sheath outer boundary condition, specifically the second of conditions 8 in the reference. The authors of the paper seem to be willing to accept this failing without qualms. The importance of this inconsistency can be seen by considering their Eq. (9) and the steps leading to it. Equation (9) is obtained by substituting (6) into (5) and integrating with respect to up to the sheath edge. The result is

$$\varepsilon_{\xi s} - \varepsilon_{\xi w} = (n\varepsilon)_w - (n\varepsilon)_s \tag{1}$$

where the subscripts indicate differentiation or the point at which the quantity is evaluated. The second of conditions 8 is $\varepsilon_s = 0$ by which it may be concluded that $(n\varepsilon)_s = 0$, giving (9). The equivalent one-dimensional situation is then introduced to evaluate $\varepsilon_{\zeta s}$ and $\varepsilon_{\zeta w}$. The solution so obtained does not satisfy the boundary conditions $\varepsilon_s = 0$. If the relations for n and ε obtained by this solution are applied to the sheath edge, it is found that $n = 1$, $\varepsilon_s = j$ and $(n\varepsilon)_s = (n\varepsilon)_w = j$. If this solution is used for both ε_s and $\varepsilon_{\zeta s}$ a completely different relation will be found for j than (14). One must conclude that Eq. (14) depends on which particular values for ε_s are selected. Since the "equivalent one-dimensional solution" gives unreasonable values for ε_s, there is no justification for using it to determine $\varepsilon_{\zeta s}$, and it is inconsistent to use this solution to determine $\varepsilon_{\zeta s}$ and the condition 8 to determine ε_s.

This Comment is not the first attempt of this type to solve the ion probe problem. I was aware of previous work by de Boer and Johnson[2] which gave similar results to those of Ref. 1 and I understand that additional references are being given in the reply to this comment. However, I would like to call attention to the analysis of Ref. 3 published several years ago in which a one-dimensional sheath solution was used which does satisfy all of the boundary conditions 8. Using the terminology of Ref. 1, the one-dimensional sheath solution of Ref. 3 is

$$\varepsilon = 2^{1/2}/A - (2^{1/2}/A)(1 + 2B - 2^{1/2}A\eta)^{1/2} \tag{2}$$

$$n = (1 + 2B - 2^{1/2}A\eta)^{-1/2} \tag{3}$$

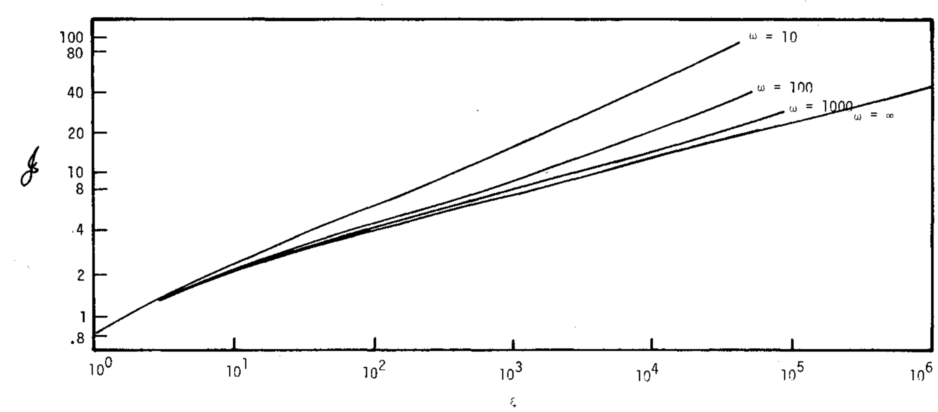

Fig. 1 Dimensionless current to flat plate probe.

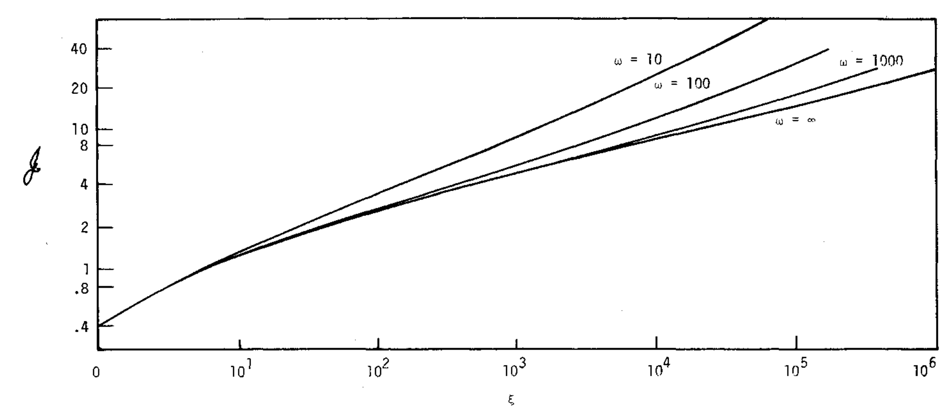

Fig. 2 Dimensionless current to conical probe.

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$$\phi = (2/3A^2)[(1+2B-2^{1/2}A\eta)^{3/2}-1]-2B/A^2+2^{1/2}\eta/A \quad (4)$$

$$j_w = (-2^{1/2}/A)(1+2B)^{-1/2}+2^{1/2}/A \quad (5)$$

$$\eta_s = 2^{1/2}B/A \quad (6)$$

$$A = [\tfrac{2}{3}(1+2B)^{3/2}-\tfrac{2}{3}-2B]^{1/2} \quad (7)$$

This solution is algebraically more complicated than the one of Ref. 1 but does satisfy the boundary conditions. j was used as the basic parameter of the sheath solution in Ref. 1, and this is also true of the above solution except that it is more convenient to introduce the auxiliary parameter B, which is a function only of j as shown by Eqs. (5) and (7). In the limit of ζ large which corresponds to A and n_s large this solution and that of Ref. 1 give the same results. This conclusion is not too surprising since in the limit of ζ large the solution for the equivalent one-dimensional situation of Ref. 1 does satisfy the boundary condition that $\varepsilon_s = 0$.

In Ref. 3, solutions were obtained not only for the limiting case of $\omega = \infty$ but for finite values of ω and for both flat plates and conical probes. The results so obtained for flat plates and conical probes are shown in Figs. 1 and 2.

References

[1] Johnson, R. A. and de Boer, P. C. T., "Theory of Ion Boundary Layers," *AIAA Journal*, Vol. 10, No. 5, May 1972, p. 664.

[2] deBoer, P. C. T. and Johnson, R. A., "Theory of Flate-Plate Ion-Density Probe," *The Physics of Fluids*, Vol. II, 1968, p. 909.

[3] Hammitt, A. G., "Negatively Charged Electrostatic Probes in High Density Plasma," Rept. 06488-6433-RO-OO, June 26, 1970, TRW Systems Group, Redondo Beach, Calif.

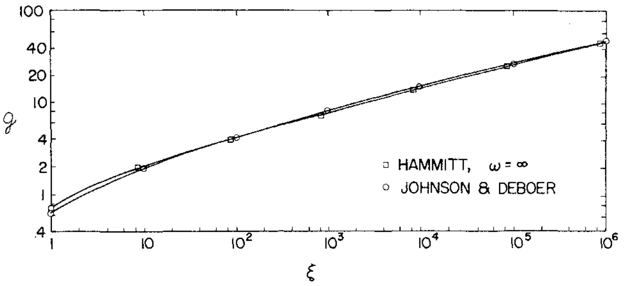

Fig. 1 Nondimensionalized current $\mathscr{J}$ as a function of ξ for flat plate probe. Squares are results of Ref. 2 for $\omega = \infty$; circles are results of Ref. 1.

Reply by Authors to A. G. Hammitt

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FIRST of all, we want to draw attention to the final two sentences of the next to last paragraph of the preceding comment by Hammitt. There, it is acknowledged that the objections raised in the comment do not apply to the downstream region (large ξ). In fact, the principal results of Refs. 1 and 2 for the flat plate are identical, and the subject of discussion is limited to "leading-edge corrections." Hammitt's implied contention that his leading-edge result constitutes an improvement over ours must be rejected as incorrect. This follows from the arguments given below, as well as from comparing the results of Refs. 1 and 2 directly (see Fig. 1). This comparison shows that the two results are in very close agreement.

Application to the leading-edge region (small and intermediate ξ) of "integral methods" such as used in Refs. 1 and 2 is not straightforward. The results obtained for this region must be regarded as empirical in nature. The difficulties go beyond the choice of a suitable approximate profile. Basically, the sheath cannot be regarded as thin near the leading edge, and a solution of the full two-dimensional equations is required. Hammitt treated this problem by "patching" his downstream solution to a solution based on assuming cylindrical symmetry around the origin. Clearly, the actual solution near the leading edge is not cylindrical, and this procedure provides no more than an estimation. In our opinion, use of the results of either Ref. 1 or 2 near the leading edge is justified mainly because of the good agreement with Dukowicz's numerical work[3] (see also Sec. II and Fig. 2 of Ref. 1).

As discussed in Ref. 5, our reasons for setting $\varepsilon_{\xi s} = -\eta_{s\xi}+j_\xi$ in Eq. (9) of Ref. 1 were as follows. Applying the equation $d\varepsilon = \varepsilon_\xi d\xi + \varepsilon_\eta d\eta = \varepsilon_\xi d\xi + n\,d\eta$ to the differential change of ε along the edge of the sheath, it is found that $\varepsilon_{\xi s} = -n_s\eta_{s\xi}+\varepsilon_{s\xi}$. Application of the boundary conditions, which already was made in setting $(n\varepsilon)_s = 0$ and $d\varepsilon_s = 0$, would yield $\varepsilon_{\xi s} = -\eta_{s\xi}$. On the other hand, use of the approximate profile of Ref. 1 gives $\varepsilon_{\xi s} = -\eta_{s\xi}+j_\xi$. For large ξ, the term j_ξ is negligibly small compared with $\eta_{s\xi}$, and the two results are equivalent. It follows that inclusion of the term j_ξ is of no consequence to the main result. The inclusion of j_ξ, which is empirical in nature, allows application of the initial condition $\eta_s = 0$ at $\xi = 0$. This obviates any need for "patching" the solution to some other result. As mentioned above, the results for $\mathscr{J}$ obtained this way agree well with the numerical work of Dukowicz.[3] When j_ξ is not retained in the expression for $\varepsilon_{\xi s}$, the solution for the current density j at small ξ is $j = C\exp(-\xi)$, where the constant C is undetermined. The "alternative choice" suggested by Hammitt, which consists of using our approximate profile also for $(n\varepsilon)_s$, is not a possible choice at all. It simply recovers the one-dimensional result, and its character is quite different from the approximation we used.

In order to establish the relation between Hammitt's work[2] and ours, we wish to point out that our theory already was available in printed form in January 1968 (Ref. 4) and also is contained in Ref. 5 which appeared in September 1969. It can easily be shown that the basic "integral" equation (34) of Ref. 2 is identical to the basic integral equation (9) of Ref. 1. While we do not disagree with Hammitt's argument that it is desirable to use an approximate profile that satisfies all of the boundary conditions, we note that Hammitt's profile does not lead to any essential improvement of the results. Apart from the comparison shown in Fig. 1, this follows by expanding the results in inverse powers of ξ, which can be compared with the exact result.[6] Both "integral methods" yield the lowest order term for $\mathscr{J}$ correctly, but the next higher order term incorrectly. The exact results[6] also show that neither the profile used by Hammitt[2] nor that used by us[1] is valid at small and intermediate ξ. As mentioned in the preceding comment, the profile used by Hammitt has the disadvantage of being more complicated algebraically than the one used by us.

In conclusion, the procedure we used in Ref. 1 to find leading-edge corrections to the main result is simpler and no less accurate than that of Ref. 2.

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